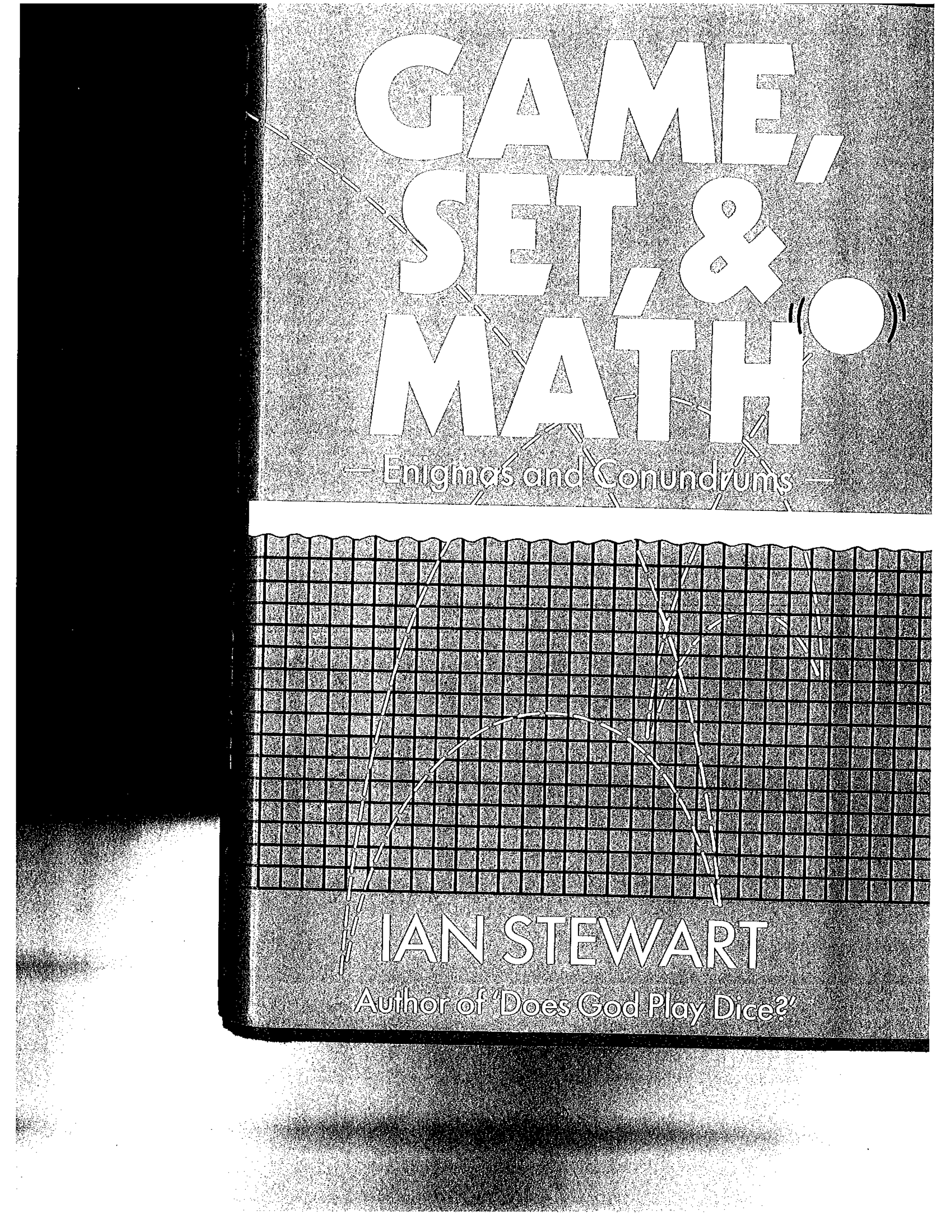
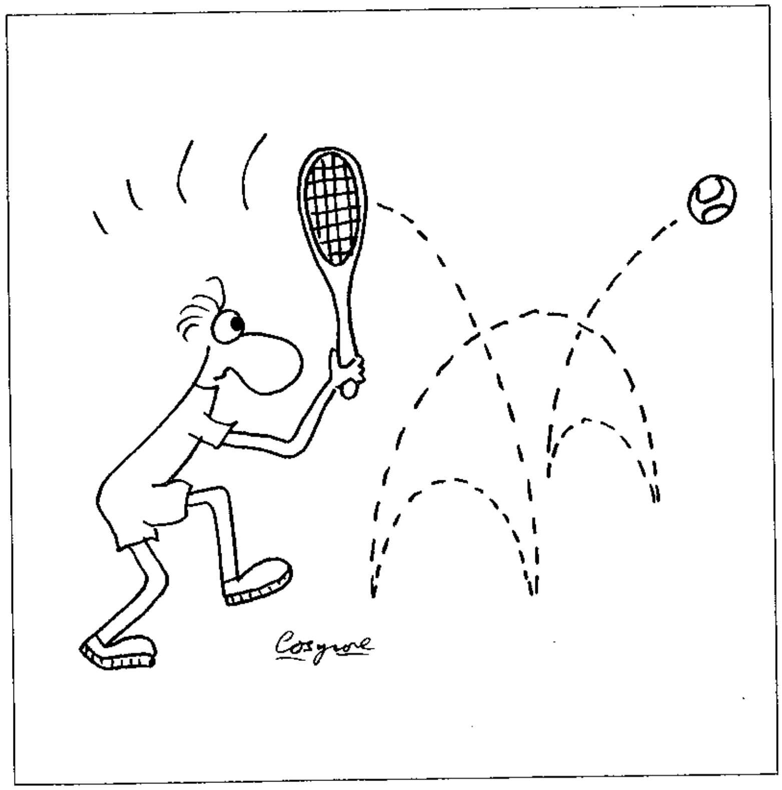

GAME, SET & MATH
— Enigmas and Conundrums —
IAN STEWART
Author of 'Does God Play Dice?'

# 2

## *The Drunken Tennis-Player*




The tennis season has started up again.

A few weeks ago, I spent the afternoon at the local tennis-club, playing an enjoyable match with my friend Dennis Racket. He won in straight sets, 6–3, 6–1, 6–2. Afterwards, as we sank a few beers in the bar, a thought struck me.

"Dennis: how come you always beat me?"

"I'm better than you, old son."

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"Yes, but you're not *that* much better. I've been keeping score and I reckon that I win one-third of the points. But I don't win one-third of the *matches!*"

"Let's face it, you don't win *any* matches against *me*." He took a quick swig at his beer. "That's because you don't win the crucial points, the ones that really matter. I mean, remember when you were leading 40–30 with the set at three games to two? You could have levelled the score at three all. Instead, you . . ."

"Served a double fault. Yes, Dennis, I know all about that. But I reckon I still win about one in three of the *crucial* points! No, there must be another explanation."

"I'd like another *beer*, that's for sure," said Dennis. "My round. I'll be right back." He heaved himself to his feet and began to negotiate his way through the crowd towards the bar. I heard him shouting over the hubbub. "Elsie! Two pints of Samuel Smith's and a packet of peanuts!" With a glass in each hand, he began to make his way back. There were so many people that he went two steps sideways for every step forwards.

Then it hit me.

*That's* why Dennis always wins!

He sat down, and I decided to share my sudden insight. "Dennis, I've worked it out! Why you always win! I was watching you coming back from the bar, and I suddenly thought: *drunkard's walk*!"

"Actually, my son, they *stagger*. Anyway, I've only had two pints!"

I hastened to reassure him that my choice of phrase was nothing personal. The drunkard's walk – less colourfully called the random walk – is a mathematical concept: the motion of a point which moves along a line, going either left or right, at random. Or on a square grid, taking steps randomly north, south, east, or west. In 1960 Frederik Pohl wrote a science fiction story called *Drunkard's Walk*, and he described it like this:

> Cornut remembered the concept with clarity and affection. He had been a second year student, and their house-master was old Wayne; the audio-visual had been a marionette drunkard, lurching away from a doll-sized lamp-post with random drunken steps in random drunken directions.

To simulate the simplest random walk, all you need is a 30 cm ruler and two coins. One coin acts as a marker, the other as a random number generator. Place the marker coin on the ruler at 15 cm. Toss the other one. If it comes down "heads", move the marker coin 1 cm to the right; if "tails", move it left (figure 2. 1).

According to probability theory, after $n$ moves you will be on average a distance $\sqrt{n}$ cm away from the middle. (Try it!) Despite this, your chances of eventually returning to the middle are 1 (certainty). On the

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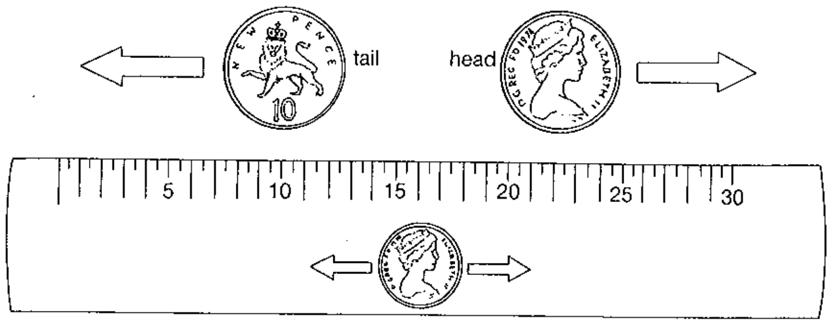
other hand, on average it takes infinitely long to get there. Random walks are subtle things. With a random walk on a square grid, you still have probability 1 of returning to the centre; but in three dimensions the probability of getting back to the centre is about 0.35. A drunkard lost in a desert will eventually reach the oasis; but an inebriated astronaut lost in space has roughly a one in three chance of getting home. Maybe they should have told ET that.




*2.1 Apparatus for a random walk.*

Years ago a probability theorist told me that the lowest dimensional space in which the chances of getting home are less than 1 is a space of $2\frac{1}{2}$ dimensions, but I've never quite worked out what he meant by that.

As you can see, mathematicians have done a lot of work on random walks. They're important. For example, they model the diffusion of molecules under random collisions in gases and liquids. And they can be used to analyse games of chance.

Such as tennis.

Dennis said he couldn't see the connection.

"But there is one," I said. "Lend me your ears and I'll try to explain why. Let's start with something simpler. Suppose Angus and Bathsheba take it in turns to toss a coin. If it comes up heads, Angus gets one point. Tails, and Bathsheba gets the point instead. Angus wins if he gets three points ahead of Bathsheba; and Bathsheba wins if she gets three points ahead of Angus. If neither has won after ten tosses, the game is a draw. Got that?"

"It's not exactly physically or intellectually challenging, this game," he muttered into his beer.

"Right then, genius: *what is Angus's chance of winning?*"

"Fifty–fifty? Oh, no, they can draw, too. One chance in three."

"I see. He can either win, draw, or lose: you think each is equally likely. Just like tossing a coin: it can either land heads, tails, or on edge, so the chance of it landing on edge is one in three."

Dennis didn't like my sarcastic tone. "All right, cleverclogs: what *is* his chance of winning?"

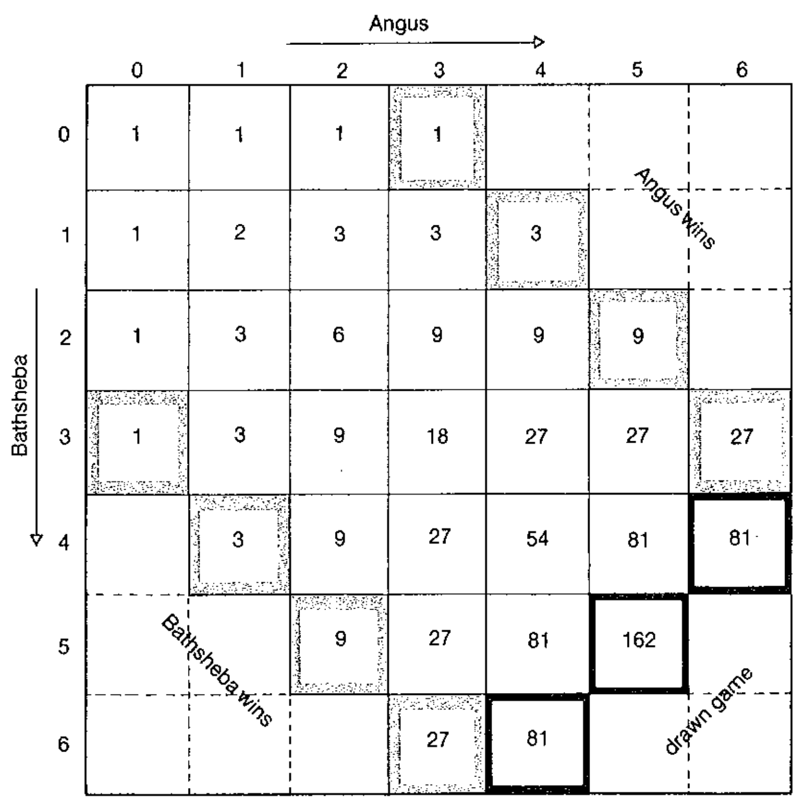
"I don't know," I said.

"Ha!"

"But if you'll pass me that napkin I'll work it out." And I started to draw a diagram (figure 2. 2).

"What's that?"

"I'm marking Angus's total score along the top, starting at 0, and Bathsheba's down the side. Then I'm going to work out how many ways




2.2 *Angus and Bathsheba toss a coin.*

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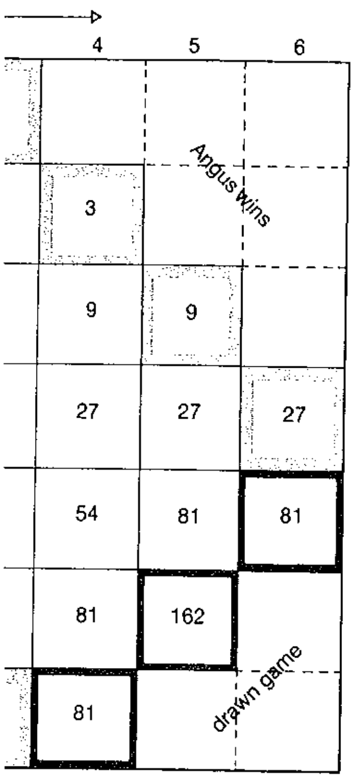




the game can reach each legal position. Then I'll count up how many of them are wins for Angus. Well, that's the gist of it, but actually I'll have to be more careful: I'll come to that in a moment."

I wrote a line of 1's along the top and down the side.

"Why all the 1's?"

"They mean – for instance – that there's only one way for Angus to go 3:0 up. He has to win all of the first three tosses."

"Ah."

"But there are *two* ways to get to a score of 1:1."

"I see that. Either Angus or Bathsheba wins the first toss, but then they lose the second."

"Exactly. In other words, the score on the previous turn is either 1:0 or 0:1 in Angus's or Bathsheba's favour – corresponding to the squares above and to the left of the 1:1 square. Each of those contains a 1, and we just add the two numbers up.

"The same method lets us work out how many ways the game can reach any given position, say *m*:*n*. The previous position was either $(m-1):n$ or $m:(n-1)$, and those are the positions above and to the left. Add them up, and write it in. Of course you have to work systematically through the possible scores. For instance, the only reason I know I can put 9 in the 3:2 square is that I've already got 3 in the 3:1 and 6 in the 2:2 positions, OK?"

"Got you."

"And you don't include squares where one player has already won, because the game stops on those. The number at 3:5, for instance, is *not* the sum of the numbers at 3:4 and 2:5, because at 2:5 Bathsheba has won and the game stops."

"It's getting complicated, old lad."

"Nonsense, you just have to be systematic and take the rules of the game into account. Now, Angus wins if the score is 3:0, 4:1, 5:2, or 6:3, and Bathsheba wins for 0:3, 1:4, 2:5, or 3:6. I'll mark those boxes with a shaded border."

"What about 7:4?"

"I said the game stops after ten tosses. That happens at scores of 4:6, 5:5, and 6:4. I'll put a heavy black border on them. *There*!"

We contemplated the diagram.

"Angus wins in 1+3+9+27 ways," said Dennis. "That's 40. He loses in 40 ways, and the game is drawn in 324. That makes 40+40+324 = 404 possibilities altogether. So his chance of winning is $\frac{40}{404}$, which is 0.0990099. About one chance in ten. That sounds unlikely to me, you must have made a mistake."

"Not quite," I said. "*You're* making a mistake. The same one as before: you're assuming each case is equally likely. But because the games go on for different numbers of turns, they *aren't* equally likely."

I bought two more beers and while we consumed them I pointed out that Probability Theory is founded on two bashic prinshiples.

1. To get the probability of a set of distinct events you add the individual probabilities.
2. To get the probability of two independent events happening in turn you multiply their probabilities together.

For instance, if you throw a fair die then the probability of each score in the range 1 to 6 is $\frac{1}{6}$, because all scores are equally likely. The probability of throwing *either* a 5 *or* a 6 is $(\frac{1}{6}) + (\frac{1}{6}) = \frac{1}{3}$. On the other hand, if you throw two dice, say a red one and a blue one, then the probability that the red one is 5 *and* the blue one is 6 is $(\frac{1}{6}) \times (\frac{1}{6}) = \frac{1}{36}$.

"To get the right answer," I told Dennis, "you just apply the rules. At each throw, Angus has a probability $\frac{1}{2}$ of winning, and so does Bathsheba. So each move one square across or down the diagram multiplies the probabilities by $\frac{1}{2}$. The chances of Angus winning 3:0 are $(\frac{1}{2}) \times (\frac{1}{2}) \times (\frac{1}{2})$, or $\frac{1}{8}$. The chances of him winning 4:1 are not $\frac{3}{8}$, but $\frac{3}{32}$, because two more tosses are involved. So his chances of winning are

$$\frac{1}{8} + \frac{3}{32} + \frac{9}{128} + \frac{27}{512}$$

which comes to $\frac{175}{512}$, or roughly 0.3418."

Dennis looked pleased with himself.

"I *told* you he had a one in three chance of winning," he said. Then he added "Ouch!" as I kicked him.

"As a check on the calculation, Dennis, you will observe that the chances of a draw are $\frac{324}{1024}$, the chances of Bathsheba winning are $\frac{175}{512}$, and the sum of the three fractions is

$$\frac{175}{512} + \frac{324}{1024} + \frac{175}{512} = 1,$$

as it must be if I haven't made any mistakes."

"You're a genius. Now, *what's all this got to do with tennis*?"

"It's the same thing, only with different rules. Tennis is a series of *points*, leading to *games*, leading to *sets*, leading to a *match*. To keep it simple, suppose Angus and Bathsheba play one *game* of tennis. On each separate point, Angus either wins or loses; and Bathsheba loses or wins. The winner of the game is the first player to get four points. Unless the score gets to three all, in which case . . ."

"Three all? Three all? What kind of tennis score is *three all*?"

"Deuce. Look, tennis has this incredibly silly scoring system that goes 15, 30, 40, *game* instead of 1, 2, 3, 4, that's all. The '40' is really '45' but

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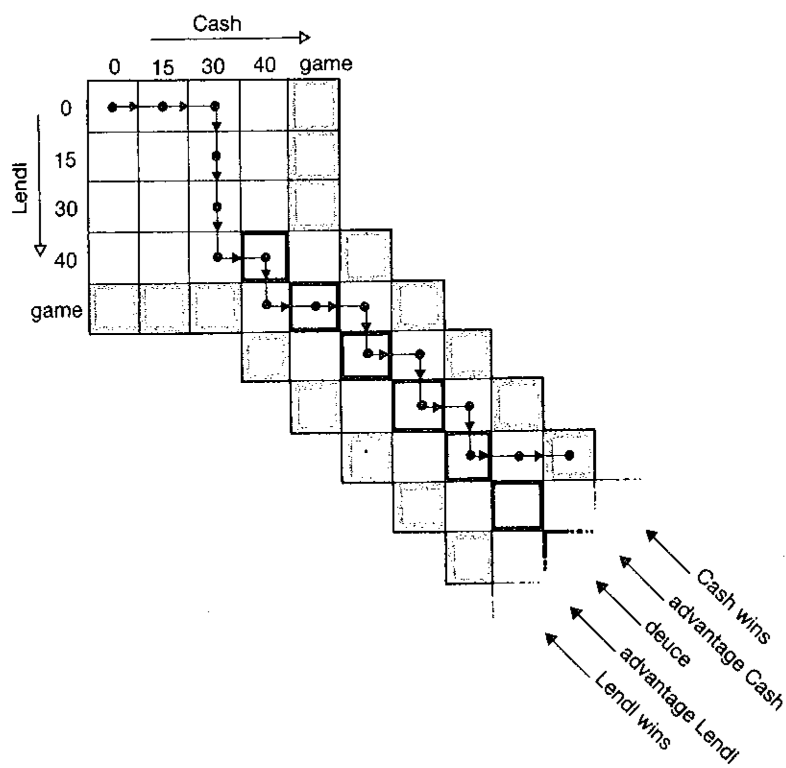
people got lazy; I suppose a game is really 60. There must have been a reason originally, but I have no idea what it was and it's just traditional now.

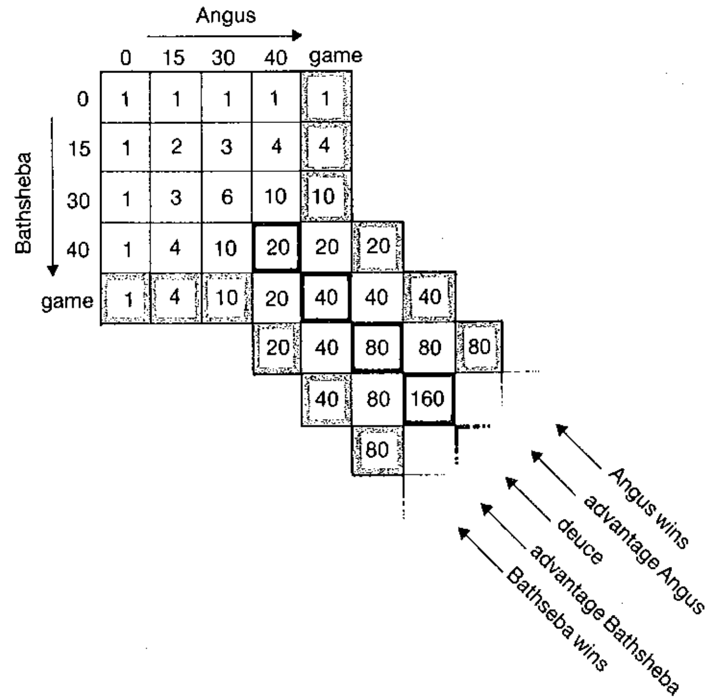
"When the score gets to deuce, the game continues until one or other player gets *two points ahead*.

"You can represent a tennis game on a diagram just like the one we drew for the coin-tossing game." I went over to the bookshelf, came back with a book of tennis scores, and picked one at random. "Look, here's the fifth game of the second set of the 1987 men's singles final at Wimbledon. Pat Cash v. Ivan Lendl. Cash was leading 3–1 in the second set and one set to love. Lendl served and lost. Here's how the scoring went." (Figure 2. 3)

"Oh, I see. It's quite clever, the way a deuce game chases off down that funny zig-zag."




2.3 *Pat Cash and Ivan Lendl play tennis: Lendl to serve.*




2.4 *Combinatorics of a game of tennis.*

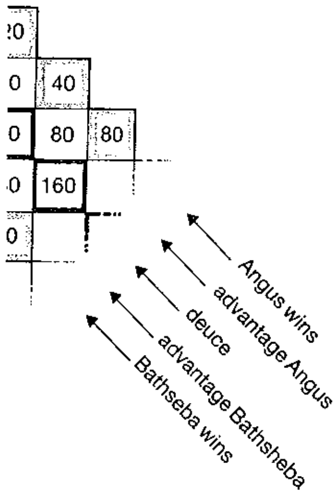
"The same thing comes up in tie-breaks too: you'll see when I get that far. In principle a game can go on forever. Of course, the chances of that are zero.

"Anyway, you can assign numbers to the squares in just the same way: each square contains the sum of the numbers immediately above and to the left, unless those squares represent the end of a game, which only matters in a deuce game (figure 2. 4).

"Of course, the chances of a given player winning a point aren't $\frac{1}{2}$ any more. Better players win points more often, as I was complaining when we first sat down. Now, to keep it simple, I'm going to assume that the chance is always the same on each point."

Dennis started to object. "But . . ."

"But players have more chance of winning if it's their serve – yes, I know that. Let's keep it simple to start with, though. The same method

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will take care of different probabilities depending who's serving, but it gets very very complicated if you do that.

"So Angus wins any given point with probability $p$, say, and loses with probability $q$, which must be equal to $1 - p$.

"Now every horizontal move is a point won by Angus, so has probability $p$, whereas a vertical move has probability $q$. For instance, the chance that Angus wins game–30 is $10p^4q^2$ because the game–30 square contains the number 10, and is four squares horizontally and two vertically away from the starting-point. His total chance of winning is $p^4 + 4p^4q + 10p^4q^2$, plus whatever happens when the game goes to deuce.

"Deuce scores complicate things a bit. But see how the numbers representing wins for Angus run down the diagonal: 10, 20, 40, 80, 160, ... doubling all the time. We have to add up an infinite series

$$10p^4q^2 + 20p^5q^3 + 40p^6q^4 + 80p^7q^5 + \dots$$

and then add on $p^4 + 4p^4q$. Now the infinite series is

$$10p^4q^2(1 + 2pq + 4p^2q^2 + 8p^3q^3 + \dots)$$

and the expression in brackets is a *geometric progression*."

"I did those at school!"

"Can you remember what the sum is?"

"No. Never saw much point to that stuff."

"$1 + r + r^2 + r^3 + \dots = \frac{1}{(1-r)}$. Provided $-1 < r < 1$, of course. *Now* you see how useful it is! Frankly, I'm amazed you play tennis so well, not knowing how to sum a geometric progression. Anyway . . . Each term is $2pq$ times the previous one, so the expression in brackets is $\frac{1}{(1-2pq)}$. That makes Angus's chance of winning exactly

$$p^4 + 4p^4q + \frac{10p^4q^2}{1-2pq}$$

Isn't that beautiful!"

"Beauty," said Dennis, "is in the eye of the beholder. Let me buy you another beer. You must be thirsty after all those calculations." He wobbled to his feet. "I know *I* am," he muttered, as he took a tentative step forward.

While he was fighting his way back to the bar, I worked out my chances of winning a game against him, assuming my chance of winning a point was one in three. That made $p = \frac{1}{3}$, $q = \frac{2}{3}$, and the formula gave me a probability of $\frac{35}{243} = 0.144$. About $\frac{1}{7}$.

"Dennis: if I have a one in three chance of winning each point, I only have a one in seven chance of winning a game! No wonder you always beat me! The rules of tennis amplify differences between players. I bet the ampifli- . . . amflipi- . . . I bet it gets even bigger when you take sets and matches into account!"

"Very likely, old son. But it's time to go home."

"Why? I was just getting . . ."

"The bar's closing."

To ease my hangover I spent the next morning working out what happens when you take sets and matches into account. The methods are just the same as those I've described, so I'll just summarize the results.

First, let's recall the rules.

In men's singles, a match consists of a maximum of five sets. A player must win at least three sets, and be two sets or more ahead of his opponent, except for a score of 3–2.

To win a set, a player must win at least six games, and be two or more games ahead. A set in the position 6–5 or 5–6 continues for a further game, and is won if the score goes to 7–5 or 5–7. If a set reaches 6–6 it proceeds to a *tie-break*, except for the fifth set in a match, in which case it continues indefinitely until one player is two games ahead.

A tie-break is much like a normal game. However, the scoring goes 0, 1, 2, ..., like the games in a set rather than the points in a game. To win, you must score at least 7, and be at least two points ahead.

Before the tie-break rule was introduced, all sets continued until one side was two games ahead. In a doubles match on 15 May 1949 F. R. Schroeder and R. Falkenburg played R. A. Gonzalez and H. W. Stewart (all of the USA) and won the first set by the margin of thirty-six games to thirty-four! The final score was 36–34, 3–6, 4–6, 6–4, 19–17, and the match took four and three-quarter hours.

You can see why the rules were changed.

The diagrams for a tie-break game, a set with or without a tie-break, and a match, are shown in figures 2.5–2.8. The corresponding formulas for probabilities of winning are shown in box 2.1. You should be able to see how they are derived from the diagrams. Capital P means "probability of winning" whatever follows it in brackets. If the play can continue indefinitely, the formula includes the sum of an infinite geometric progression.

The rules for women's singles are slightly different. A match can be won either two sets to love or two sets to one. Tie-breaks are played in every set. You might like to carry out this analysis yourself.

By fitting all the formulas together you can, in principle, write down an explicit expression for the probability of winning a tennis-match. I've indicated with arrows in box 2.1 how to do this: substitute for $p$ the expression in the box at the tail of the arrow, and one minus this for $q$. I haven't actually carried this procedure out, because the result would be *enormous*. Each single $p$ or $q$ in one formula becomes an entire expression from the previous formula, and the complications become horrendous.

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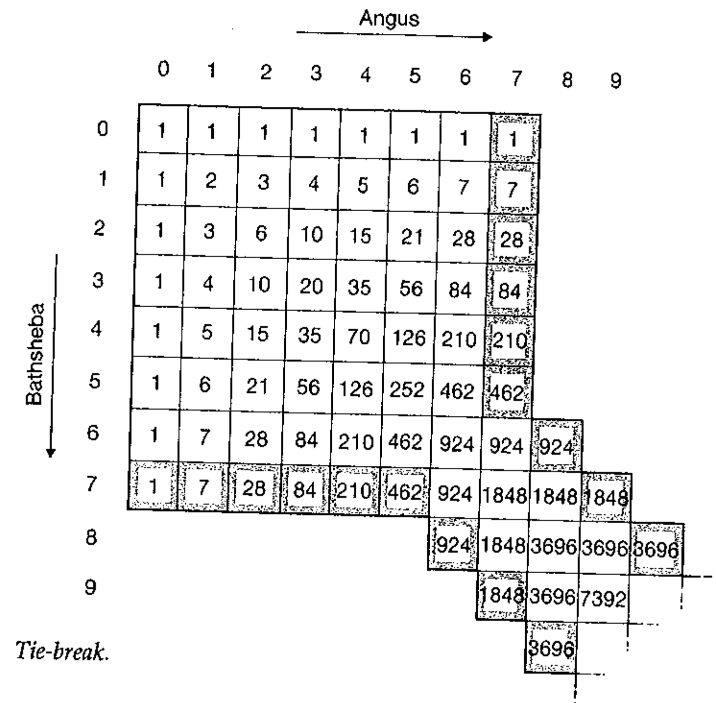
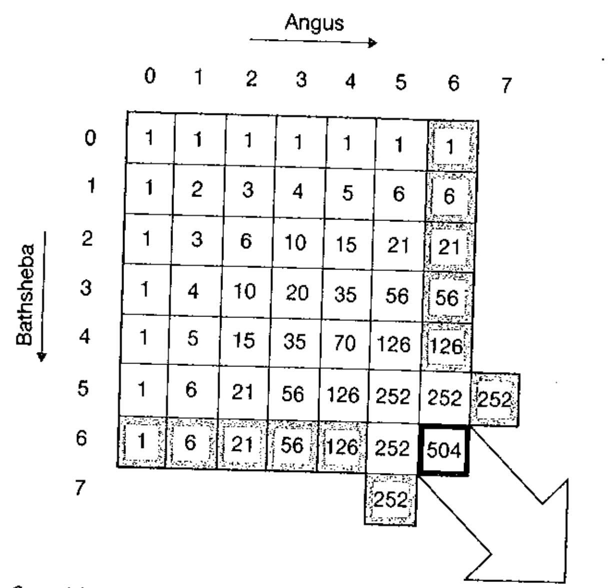
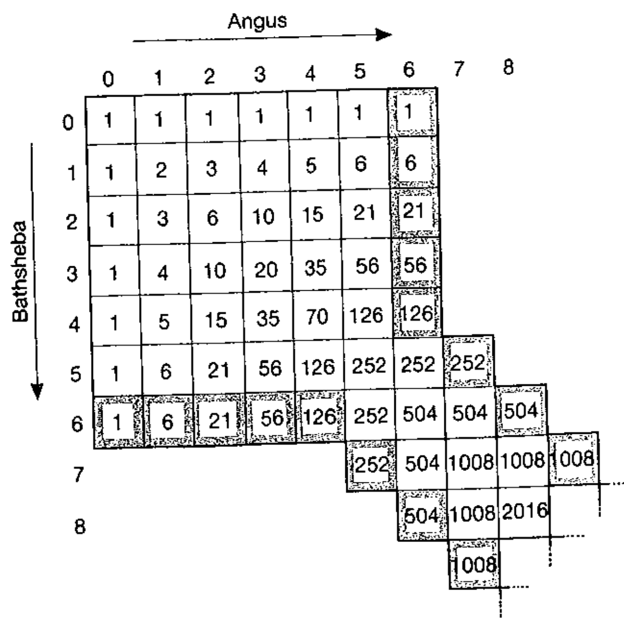


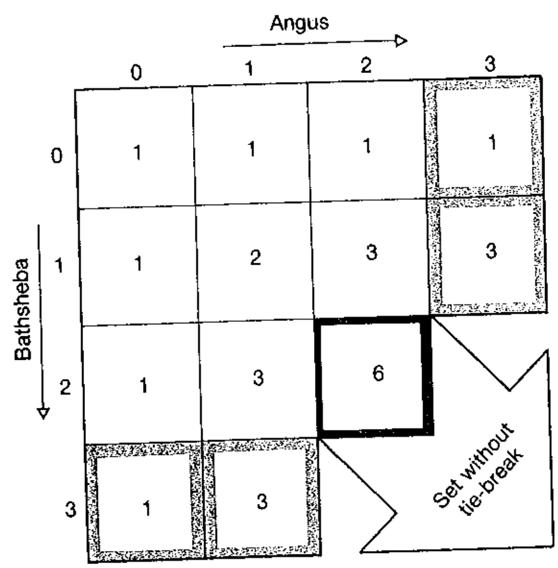

*2.5 Tie-break.*



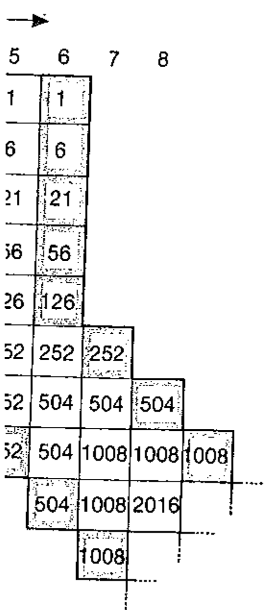

*2.6 Set with tie-break.*

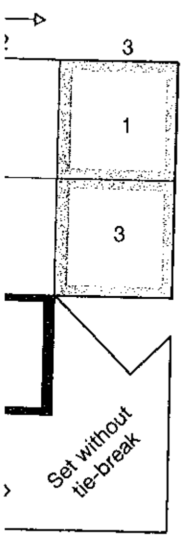



2.7 *Set without tie-break.*




2.8 *Match.*








**Box 2.1 Game, set, and match: probabilities of winning**

Game

$p = \mathrm{P}(\text{point}),\, q = 1 - p$

$$\boxed{p^4 + 4p^4q + \frac{10p^4q^2}{1-2pq}}$$

Tie-break

$p = \mathrm{P}(\text{point}),\, q = 1 - p$

$$\boxed{p^7 + 7p^7q + 28p^7q^2 + 84p^7q^3 + 210p^7q^4 + \frac{462p^7q^5}{1-2pq}}$$

Set with tie-break

$p = \mathrm{P}(\text{game}),\, q = 1 - p$

$$\boxed{p^6 + 6p^6q + 21p^6q^2 + 56p^6q^3 + 126p^6q^4 + 252p^7q^5 + 504p^6q^6\mathrm{P}\,(\text{tie-break})}$$

Set without tie-break

$p = \mathrm{P}(\text{game}),\, q = 1 - p$

$$\boxed{p^6 + 6p^6q + 21p^6q^2 + 56p^6q^3 + \frac{126p^6q^4}{1-2pq}}$$

Match

$p = \mathrm{P}(\text{set with tie-break}),\, q = 1 - p$

$$\boxed{p^3 + 3p^3q + 6p^2q^2\mathrm{P}\,(\text{set without tie-break})}$$

However, you can substitute values from one formula to the next, and I've shown what happens in figure 2.9. This gives a table, and a graph, of the probability of winning a men's singles match if your probability of winning any individual point is $p$.

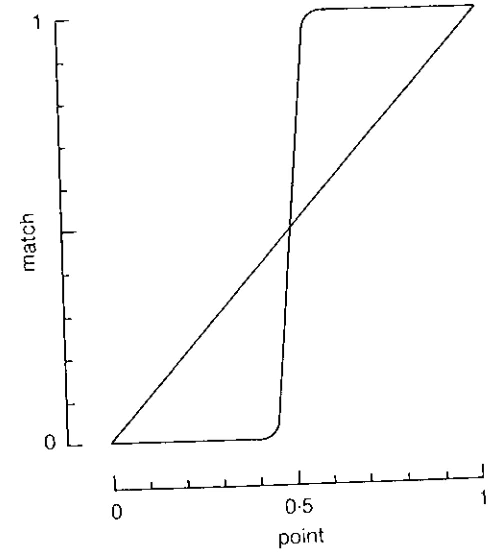
I showed all this to Dennis the next evening.

"Should Bathsheba be playing men's singles?" he objected.

"She's very liberated. She's thinking of changing her name to Boris. Now, shut up and listen. Observe that the graph is very flat at each end but rises extremely steeply in the middle. With a probability of more than 0.6 of winning each point, your chance of winning the game is nearly 1. *The rules of tennis favour the better player.*"

He stared at me over his beer, perplexed. "But they should, shouldn't they? I mean, the better player ought to have the better chance of winning."

| probability of winning | |
|---|---|
| point | match |
| 0 | 0 |
| 0·1 | 0 |
| 0·2 | $10^{-22}$ |
| 0·3 | $4{\cdot}5 \times 10^{-11}$ |
| 0·4 | $4{\cdot}4 \times 10^{-4}$ |
| 0·5 | 0·5 |
| 0·6 | 0·9995 |
| 0·7 | 0·9999 |
| 0·8 | 0·9999 |
| 0·9 | 1 |
| 1·0 | 1 |




2.9 *Calculating the winning probability.*

"True."

"But you say all this depends on the assumption that the probability of winning a point is always the same. That's not very realistic."

"You're referring to the advantage of serving."

"Right! When a player is serving, he stands a much better chance of winning the point than when he's receiving – present company excepted, of course."

"Hmph."

"Shows how important the serve is."

"I could redo the calculations . . ."

"Not on my account. I've got the message. You can apply probability theory to tennis." He sank mockingly to his knees and bowed his head to the floor. "I believe, I believe!"

I ignored his antics. "Mmm, but it might be interesting . . . You see, the way the scoring amplifies any advantage means that each player has a chance rather close to 1 of winning his service game – provided his chance of winning a point is above $\frac{1}{2}$. That tends to act the *opposite* way, which evens the game out again! Where's that pencil? . . ."

"Hang on," said Dennis, heaving himself back into his chair. "Before you cover the tablecloth with algebra, answer me one thing. On this theory of yours, what chance do you have of beating *me*?"

"Well," I said, "according to my calculations, if I have a $\frac{1}{3}$ chance of winning a point against you, my chance of winning a match is 0.000000027, or about one in thirty-seven million."

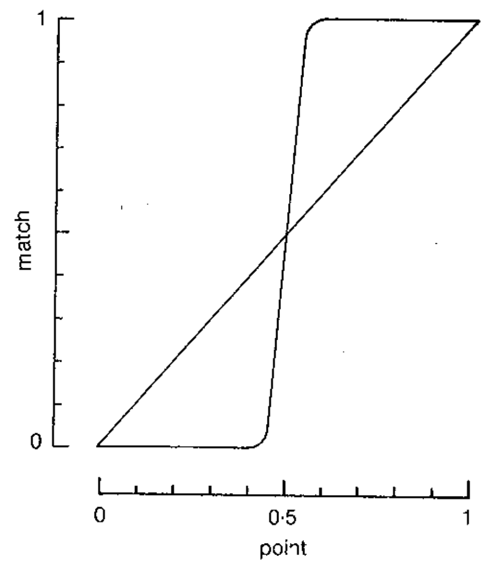
"I'd leave the theory just as it is," he said. "It looks perfect to me."

## ANSWERS

The probability of winning a set in women's singles tennis is $p^2 + 2p^2q$, where $p = \mathrm{P}(\text{set with tie-break})$ and $q = 1 - p$.

The graph of how this varies with the probability $p = \mathrm{P}(\text{point})$ is shown in figure 2.10.

| probability of winning | |
|---|---|
| point | match |
| 0 | 0 |
| 0·1 | $10^{-29}$ |
| 0·2 | $1{\cdot}4 \times 10^{-15}$ |
| 0·3 | $8{\cdot}4 \times 10^{-8}$ |
| 0·4 | $3{\cdot}9 \times 10^{-3}$ |
| 0·5 | 0·5 |
| 0·6 | 0·9961 |
| 0·7 | 0·9999 |
| 0·8 | 0·9999 |
| 0·9 | 1 |
| 1·0 | 1 |




*2.10 Calculating the winning probability for women's singles.*

## FURTHER READING

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